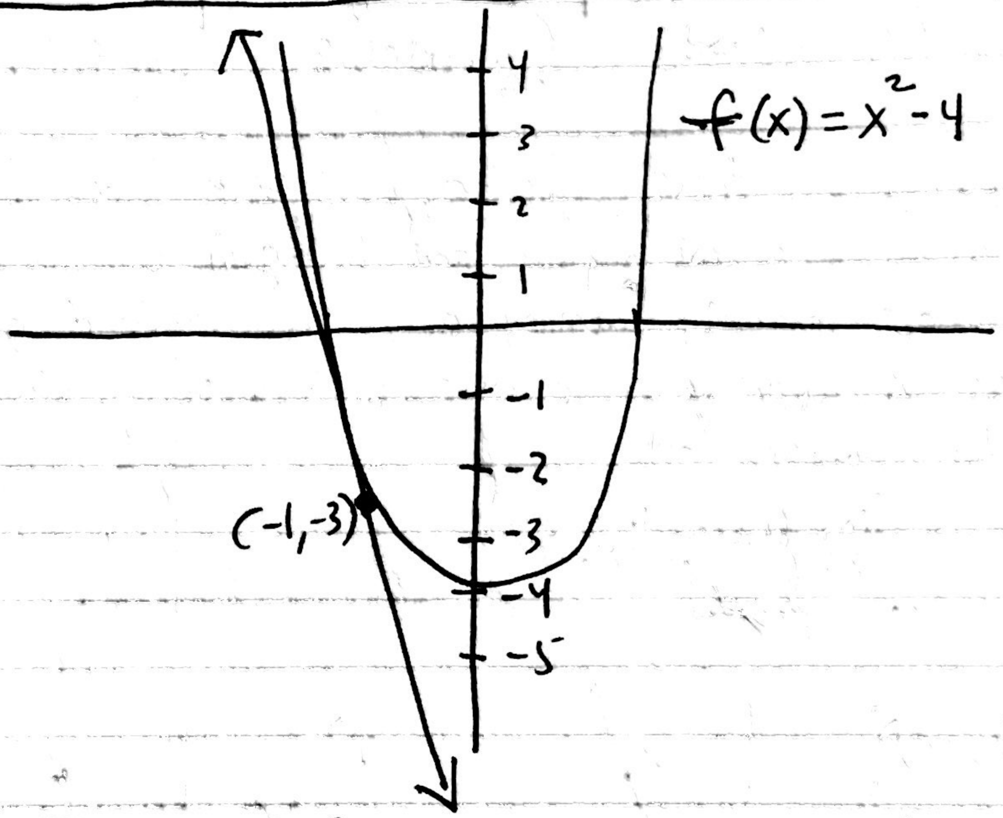
Chapter 2: The derivative and the Tangent Line Problem

Definition of Tangent Line with Slope m

Slope of tangent line $(c, f(c))$ (derivative):

$$f'(x) = m = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$



The slope of f at any point $(c, f(c))$ is
$m = 2c$ because $f'(x) = 2x$

Derivative - 1st

slope of tangent line at any given point

eg. $f(x) = x^2 - 4$ $\quad f'(x) = 2x$

at point $(-1,-3)$, the slope of the tangent line is -2

Notations for Derivatives

$f'(x) \quad \frac{dy}{dx} \quad y' \quad \frac{d}{d(x)}[f(x)] \quad D_x[y]$

Finding the Derivative by the Limit Process

$f(x) = x^3 + 2x$

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ ← definition of derivative

$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 + 2(x+\Delta x) - (x^3+2x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{3x^2 \cancel{\Delta x} + 3x(\Delta x)^{\cancel{2}} + (\Delta x)^{\cancel{3}\,2} + 2\cancel{\Delta x}}{\cancel{\Delta x}}$

$= \lim_{\Delta x \to 0} [3x^2 + 3x\Delta x + (\Delta x)^2 + 2]$

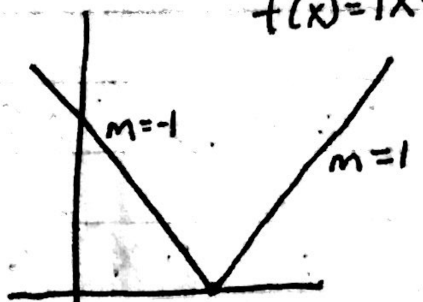
$= 3x^2 + 2$

Differentiability and Continuity

If f is differentiable at $x=c$, then f is continuous at $x=c$

Graph with sharp turn

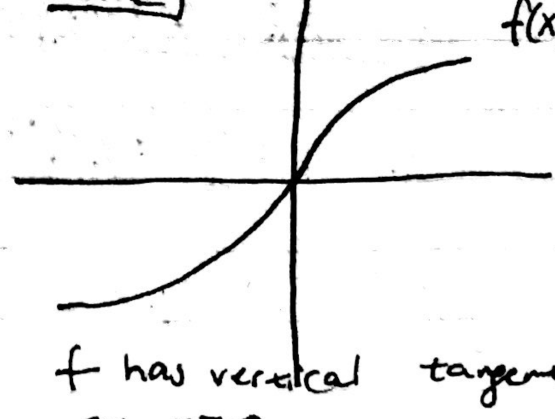
$f(x) = |x-2|$



derivatives from left and right are not equal

Graph with vertical tangent line

$f(x) = x^{1/3}$



f has vertical tangent line at $x=0$

The Power Rule

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0

a) $f(x) = x^3$ $\quad$ $f'(x) = 3x^2$

b) $g(x) = \sqrt[3]{x}$ $\quad$ $g'(x) = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3}$
$= \frac{1}{3x^{2/3}}$

c) $y = \frac{1}{x^2} = x^{-2}$ $\quad$ $y' = -2x^{-3} = \frac{-2}{x^3}$

The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also differentiable

$$\frac{d}{dx}[c\, f(x)] = c\, f'(x)$$

a) $y = \frac{2}{x} = 2x^{-1}$ $\quad$ $y' = -2x^{-2} = \frac{-2}{x^2}$

d) $y = \frac{1}{2\sqrt[3]{x^2}} = \frac{1}{2}\ x^{-\frac{2}{3}}$ $\quad$ $y' = \frac{-2}{6}x^{-\frac{5}{3}} = -\frac{1}{3x^{5/3}}$

The Sum and Difference Rules

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$$

Derivatives of Sine and Cosine functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

e.g.

a) $y = 2\sin x$ $\qquad y' = 2\cos x$

b) $y = 2\cos x$ $\qquad y' = -2\sin x$

Applications of Calculus in Physics

position function:

$$s(t) = \frac{1}{2}gt^2 + V_0 t + S_0$$

velocity function:

$$s'(t) = V(t) = gt + V_0$$

speed is the absolute value of velocity:

speed is a vector, velocity is scalar

The Product Rule

$$\frac{d}{dx}[f(x)\,g(x)] = f'(x)\,g(x) + g'(x)\,f(x)$$

e.g. $h(x) = (3x - 2x^2)(5+4x)$

$$h'(x) = (3x-2x^2)'(5+4x) + (5+4x)'(3x-2x^2)$$

$$= (3-4x)(5+4x) + (\cancel{3}4)(3x-2x^2)$$

$$= (15-8x-16x^2) + (12x-8x^2)$$

$$= -24x^2 + 4x + 15$$

The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - g'(x)\,f(x)}{(g(x))^2} \qquad g(x) \neq 0$$

e.g. $y = \frac{5x-2}{x^2+1}$

$$y' = \frac{(5x-2)'(x^2+1) - (x^2+1)'(5x-2)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)5 - (2x)(5x-2)}{(x^2+1)^2}$$

$$= \frac{(5x^2+5) - (10x^2-4x)}{(x^2+1)^2}$$

$$= \frac{-5x^2+4x+5}{((x^2+1)^2}$$

Derivatives of Trig functions

$\frac{d}{dx}[\tan x] = \sec^2 x$ $\qquad$ $\frac{d}{dx}[\cot x] = -\csc^2 x$

$\frac{d}{dx}[\sec x] = \sec x \tan x$ $\qquad$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Notations of second derivatives

y'' $\qquad$ $f''(x)$ $\qquad$ $\frac{d^2y}{dx^2}$ $\qquad$ $\frac{d^2}{dx^2}[f(x)]$ $\qquad$ $D_x^2[y]$

The Chain Rule

If $y=f(u)$ is a differentiable function of u and $u=g(x)$ is a differentiable function of x, then $y=f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\, g'(x)$$

e.g. $y=(x^2+1)^3$

$y' = 3(x^2+1)^2(2x)$ $\qquad \Leftarrow \quad y = [(x^2+1)^3]'(x^2+1)'$

$= 6x(x^2+1)^2$

The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number then,

$$\frac{dy}{dx} = n\,[u(x)]^{n-1}\ \frac{du}{dx}$$

or equivalently

$$\frac{d}{dx}[u^n] = n\,u^{n-1}\,u'$$

e.g. $f(x) = (3x-2x^2)^3$

Let $u = 3x-2x^2$

$$f(x) = (3x-2x^2)^3 = u^3$$

$$f'(x) = 3(3x-2x^2)^2\,(3x-2x^2)'$$

$$= 3(3x-2x^2)^2(3-4x)$$

Trigonometric Functions and the Chain Rule

$$\frac{d}{dx}[\sin u] = (\cos u)\,u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)\,u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\,u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)\,u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\,u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)\,u'$$

e.g. $f(x) = \sin(3x^2)$

$$f'(x) = (\cos 3x^2)\,6x$$

Explicit Form:
$y = 3x - 2$

Implicit Form:
$3x + y = -2$

The Official Guide to Implicit Differentiation

1) Differentiate both sides of the equation with respect to x
2) Collect all terms involving $\frac{dy}{dx}$ on the left of the equation and move al other terms to the right side
3) Factor $\frac{dy}{dx}$ out of the left side of the equation
4) Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by the left hand factor that does not contain $\frac{dy}{dx}$

e.g. $\frac{dy}{dx}\left[y^3 + y^2 - 5y - x^2 = 4\right]$

$$3y^2 \frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

$$3y^2 \frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

Related Rates!!

Guidelines For Solving Related Rates Problems

1. Identify all given quantities and quantities to be determined. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change are either given or are to be determined.
3. Using the chain Rule, implicitly differentiate both sides of the equation (with respect to time)
4. After completing step 3, substitute into the resulting equation all know values for the variables and their rates of change. Then solve for the required rate of change
5. Rember the units

Example:

A pebble is dropped into a pond, causing ripples in the form of concentric circles.

The radius r of the outer ripple is increasing at a constant rate of 1 ft/s.

When the radius is 4 ft, at what rate is the total area A of the water changing?

Variables

$\frac{dr}{dt} = 1 \text{ ft/s}$ $\quad r=4 \quad$ $\frac{dA}{dt} = ?$ $\quad A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$= 2\pi(4)(1)$$

$$\frac{dA}{dt} = 8\pi \text{ ft}^2/\text{s}$$